Jordan Blocks of Unipotent Elements in Spin Groups

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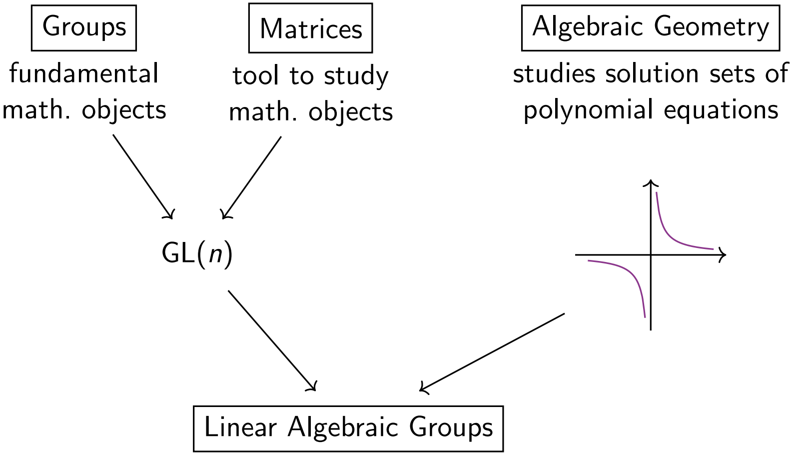
$$\begin{pmatrix} \boxed{1} & & & & \\ & \boxed{1} & & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

The Setting


Groups
fundamental
math. objects
Matrices
tool to study
math. objects
Algebraic Geometry
studies solution sets of
polynomial equations
GL(n)
Linear Algebraic Groups

Linear Algebraic Groups

Throughout: K algebraically closed field with $\mathrm{char}(K) \neq 2$, $n \in \mathbb{Z}_{>0}$, $\mathrm{GL}(n) := \mathrm{GL}(n, K)$

Definition

An affine variety (over K) is the common zero locus in K^n of a set of polynomials $S \subseteq K[X_1, \ldots, X_n]$.

Definition

A group G is a linear algebraic group if it is an affine variety and the group operations $G \times G \to G,\ (x, y) \mapsto xy$ and $G \to G,\ x \mapsto x^{-1}$ are given by polynomial equations in the coordinates.

- Analogous to Lie groups, topological groups
- Methods from both group theory and algebraic geometry available, giving powerful theory

Linear Algebraic Groups

Example

- $(K, +)$ is zero locus of $0 \in K[X]$ and a linear algebraic group
- $\mathrm{GL}(n) \cong \{(A, y) \in K^{n \times n} \times K \mid \det A \cdot y = 1\}$ is linear algebraic group

Term "linear" refers to the following fact:

Theorem

Every linear algebraic group is isomorphic to a linear algebraic group contained in $\mathrm{GL}(r)$ *for some* $r \in \mathbb{Z}_{>0}$.

Linear Algebraic Groups

Alternative Definition

A linear algebraic group (over K) is a subgroup of $\mathrm{GL}(n)$ that is defined by polynomial equations for the matrix entries.

Example

- Special linear group $\mathrm{SL}(n) = \{A \in \mathrm{GL}(n) \mid \det A = 1\}$
- Special orthogonal group

$$\mathrm{SO}(n) = \{A \in \mathrm{GL}(n) \mid AA^\top = \mathrm{Id}_n, \det A = 1\}$$

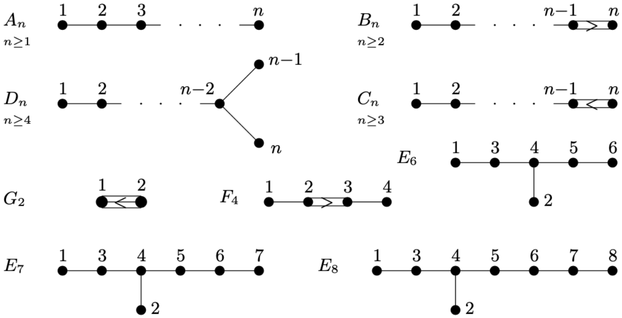
Have connections to many areas of algebra, e.g. number theory and finite group theory

Classification

- Every linear algebraic group can be "split up" into a finite part, a solvable part and a semisimple part
- Every semisimple group is product of simple linear algebraic groups

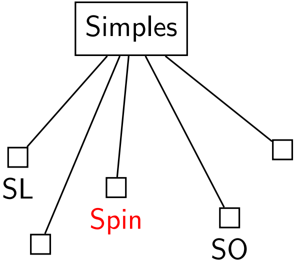
▶ Simples are building blocks for semisimples

- Simples can be classified by combinatorial data (Dynkin Diagrams)!



Spin Groups

- Among simple linear algebraic groups is family of spin groups Spin(n)
  - Important objects of study
- "Problem": not constructed as a subgroup of GL, but abstractly
- To study Spin(n), use representations



Spin Representation

Definition

A (matrix) representation of a group G is a group homomorphism $G \to \mathrm{GL}(m)$ for some $m \in \mathbb{Z}_{>0}$.

- Allows to study groups via matrices and linear algebra which we know well!
- In case of spin groups, study the spin representation

$$\mathrm{srep}_n \colon \mathrm{Spin}(n) \to \mathrm{GL}(2^{\lfloor \frac{n}{2} \rfloor})$$

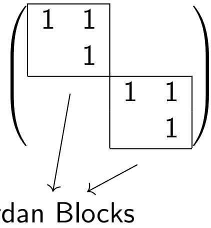
that arises naturally

Unipotent Elements

Definition

$A \in \mathrm{GL}(n)$ is called unipotent if all its eigenvalues are 1.

- Unipotent elements play important role in structure theory of linear algebraic groups
- Are interested in their Jordan normal form because it encodes a lot of information



Main Problem of my Thesis

Main Problem, 1st Formulation

For $u \in \mathrm{Spin}(n)$ unipotent, find the Jordan normal form of $\mathrm{srep}_n(u)$

- Observation: If $A, B \in \mathrm{GL}(m)$, then A and BAB^{-1} have same Jordan normal form
- ▶ Suffices to consider conjugacy classes $\{xux^{-1} \mid x \in \mathrm{Spin}(n)\}$ of unipotent elements u

Main Problem

For $C \subseteq \mathrm{Spin}(n)$ unipotent class, find the Jordan normal form of $\mathrm{srep}_n(C)$

Construction of Spin(n)

$V := K^n$, $Q := \sum_{i=1}^n X_i X_{n+1-i}$ quadratic form on V.
$T(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$ tensor algebra of V

Definition

The Clifford algebra of Q is $\mathrm{Cliff}(n) := T(V)/\langle v \otimes v - Q(v) \mid v \in V \rangle$

Note: $V \subseteq \mathrm{Cliff}(n)$ generates $\mathrm{Cliff}(n)$, $v^2 = Q(v) \in \mathrm{Cliff}(n)$

Definition

$\mathrm{Spin}(n) := \{x \in \mathrm{Cliff}(n)^\times \mid xVx^{-1} \subseteq V, \text{ plus some normalizing condition}\}$

Example (Low dimensions)

$\mathrm{Spin}(1) = \{\pm 1\}, \quad \mathrm{Spin}(2) \cong K^\times, \quad \mathrm{Spin}(3) \cong \mathrm{SL}(2).$

Construction of Spin(n)

Definition

Spin(n) $:= \{x \in \mathrm{Cliff}(n)^\times \mid xVx^{-1} \subseteq V, \text{ plus some normalizing condition}\}$

- For $x \in \mathrm{Spin}(n)$ let $\varphi_x \colon V \to V,\ v \mapsto xvx^{-1}$. Get exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Spin}(n) & \xrightarrow{\varphi} & \mathrm{SO}(n) & \longrightarrow & 1 \\ & & & & x & \longmapsto & \varphi_x & & \end{array}$$

- Closely relates Spin(n) and SO(n)!
- Have generating system:

$$\mathrm{Spin}(n) = \langle uv \mid u, v \in V, Q(u) = Q(v) = -1 \rangle.$$

Spin Representation

- Depending on parity of n, $\mathrm{Cliff}(n)$ only has 1 resp. 2 irreducible representations
- Spin representation $\mathrm{srep}_n\colon \mathrm{Spin}(n) \to \mathrm{GL}(2^{\lfloor \frac{n}{2} \rfloor})$ is restriction of irreducible representation of $\mathrm{Cliff}(n)$ (independent of choice)
- srep_n can be computed on generating system of $\mathrm{Spin}(n)$

Example ($n = 2$)

Here, can give full description:

$$\mathrm{srep}_2\colon \mathrm{Spin}(2) \cong K^\times \to \mathrm{GL}(2),\ t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

For bigger n more complicated and less explicit!

Determining the Unipotent Classes of Spin(n)

- Group homomorphism $\varphi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ induces bijection between unipotent classes of Spin(n) and SO(n)
- Unipotent conjugacy classes of SO(n) well-known! Are described in terms of Jordan normal forms
- Finitely many and can easily be computed

$$\mathrm{Spin}(n) \xrightarrow{\varphi} \mathrm{SO}(n)$$

$$\{\text{unip. classes of } \mathrm{Spin}(n)\} \xleftrightarrow{1:1} \{\text{unip. classes of } \mathrm{SO}(n)\}$$

easy!

Situation for very small n

Example ($n = 1$)

Only one unipotent class in $\mathrm{Spin}(1)$, the class of the identity element e. Have $\mathrm{srep}_1(e) = (1) \in \mathrm{GL}(1)$ ✓

Example ($n = 2$)

Again only the class of $e \in \mathrm{Spin}(2)$. Here, $\mathrm{srep}_2(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ ✓

- For $n \geq 3$ more than one unipotent class
- Cannot compute srep_n explicitly for all classes

Approach to the Problem

- For $1 \leq r < n$ established inclusion $\mathrm{Spin}(r) \subseteq \mathrm{Spin}(n)$ and group homomorphism

$$\beta\colon \mathrm{Spin}(r) \times \mathrm{Spin}(n-r) \to \mathrm{Spin}(n)$$

- Idea to tackle Main Problem: Restrict spin representation and use induction!

- Two questions arising:
  1. How does restriction of srep_n look like?
  2. Which unipotent classes are in the image of β?

Q1: How does restriction of srep_n look like?

Known result:

Theorem (Meinrenken)

$$\mathrm{srep}_n\big|_{\mathrm{Spin}(n-1)} = \begin{cases} \mathrm{srep}_{n-1} \oplus \mathrm{srep}_{n-1}, & n \text{ even}, \\ \mathrm{srep}_{n-1}, & n \text{ odd}. \end{cases}$$

- Proof relies on representation theory of Clifford algebras
- Approach can be adapted to the situation

$$\beta\colon \mathrm{Spin}(r) \times \mathrm{Spin}(n-r) \to \mathrm{Spin}(n)!$$

Q1: How does restriction of srep_n look like?

General statement needs:

Definition

Let $A \in \mathrm{GL}(s)$, $B \in \mathrm{GL}(t)$. The Kronecker product of A and B is

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1s}B \\ \vdots & \ddots & \vdots \\ a_{s1}B & \cdots & a_{ss}B \end{pmatrix} \in \mathrm{GL}(st).$$

Definition

Let $\rho\colon G \to \mathrm{GL}(s)$, $\sigma\colon H \to \mathrm{GL}(t)$ representations of groups G, H. Then

$$\rho \otimes \sigma\colon G \times H \to \mathrm{GL}(st),\ (g,h) \mapsto \rho(g) \otimes \sigma(h)$$

is a representation of $G \times H$, the tensor product of ρ and σ.

Q1: How does restriction of srep_n look like?

Restriction Theorem (A.)

Let $1 \leq r < n$ and $\beta\colon \mathrm{Spin}(r) \times \mathrm{Spin}(n-r) \to \mathrm{Spin}(n)$ as before. Then

$$\mathrm{srep}_n \circ \beta = \begin{cases} (\mathrm{srep}_r \otimes \mathrm{srep}_{n-r}) \oplus (\mathrm{srep}_r \otimes \mathrm{srep}_{n-r}), & n \text{ even, } r \text{ odd,} \\ \mathrm{srep}_r \otimes \mathrm{srep}_{n-r}, & \text{else} \end{cases}$$

- For $r = 1$ retrieve old result
- Can be refined for irreducible constituents of srep_n
- Thus in $\operatorname{im} \beta \subseteq \mathrm{Spin}(n)$ can use lower-dimensional results! Kronecker product easy to compute

Q2: Which unipotent classes are in the image of β?

- Have commutative diagram with simple map on the SO-side

$$
\begin{array}{ccc}
\mathrm{Spin}(r) \times \mathrm{Spin}(n-r) & \xrightarrow{\beta} & \mathrm{Spin}(n) \\
\downarrow & & \downarrow \varphi \\
\mathrm{SO}(r) \times \mathrm{SO}(n-r) & \longrightarrow & \mathrm{SO}(n) \\
(A, B) & \longmapsto & \begin{pmatrix} A & \\ & B \end{pmatrix}
\end{array}
$$

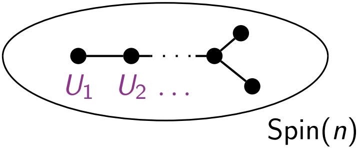
- Allows to check question for classes of SO where this is easy!

- Turns out: Except for one unipotent class, a member of every class is in the image of beta!

- Remains to compute Jordan blocks for exceptional class

Dealing with the Exceptional Class

- Determined root subgroups U_i of spin group which reveal a lot of its structure
- Can be described using generating system of Spin(n)



- Exceptional class has explicit description in terms of the U_i and therefore in terms of the generators of Spin(n)

▶ Jordan blocks of exceptional class can be computed directly!

Computation of the Jordan Blocks of Unipotent Classes

- $n = 1, 2$ ✓
- $n \geq 3$: Jordan blocks for all unipotent classes except one can be computed inductively using Restriction Theorem
- Blocks of the exceptional class can be computed directly using knowledge on root subgroups

Gives recursive algorithm. Has been implemented in the Computer Algebra System GAP

Results – Notation

- Recall: all eigenvalues of unipotent elements are 1. Same holds for image under spin representation
- If a block of size s occurs with multiplicity m, write s^m, e.g.

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \longrightarrow 1^2, 3$$

- For simplicity, unipotent classes have no specific labels

Results in Low Dimensions

n	Class	Jordan blocks
1	C_{11}	1
2	C_{21}	1^2
3	C_{31}	1^2
	C_{32}	2
4	C_{41}	1^4
	C_{42}	2^2
	C_{43}	1^2, 2
	C_{44}	1^2, 2

n	Class	Jordan blocks
5	C_{51}	1^4
	C_{52}	2^2
	C_{53}	1^2, 2
	C_{54}	4
6	C_{61}	1^8
	C_{62}	2^4
	C_{63}	1^4, 2^2
	C_{64}	4^2
	C_{65}	1^2, 3^2

Dependence on Characteristic

Blocks only depend on $\mathrm{char}(K)$ up to a certain extent:

Theorem (A.)

For each n there exists a minimal bound $B_n \in \mathbb{Z}_{\geq 0}$ such that the Jordan blocks for $\mathrm{Spin}(n)$ in any characteristic $\geq B_n$ are the same as the ones in characteristic 0.

Have $B_n = 0$ precisely for $n \leq 8$.

Results in Low Dimensions

n	Class	Jordan blocks
1	C_{11}	1
2	C_{21}	1^2
3	C_{31}	1^2
	C_{32}	2
4	C_{41}	1^4
	C_{42}	2^2
	C_{43}	1^2, 2
	C_{44}	1^2, 2

n	Class	Jordan blocks
5	C_{51}	1^4
	C_{52}	2^2
	C_{53}	1^2, 2
	C_{54}	4
6	C_{61}	1^8
	C_{62}	2^4
	C_{63}	1^4, 2^2
	C_{64}	4^2
	C_{65}	1^2, 3^2

Block Structure

Theorem (Malle–A.)

For $n \geq 7$, every unipotent class has at least two Jordan blocks.

n	Class	Jordan blocks
7	C_{71}	1^8
	C_{72}	2^4
	C_{73}	$1^4, 2^2$
	C_{74}	4^2
	C_{75}	$1^2, 3^2$
	C_{76}	$1, 2^2, 3$
	C_{77}	$1, 7$

Theorem (A.)

- *If $n \equiv 0, 1, 7 \mod 8$, then even sized blocks occur with an even multiplicity.*
- *If $n \equiv 3, 4, 5 \mod 8$, then odd sized blocks occur with an even multiplicity.*

Summary

Context

- Linear algebraic groups combine group theory with algebraic geometry; one of their building blocks are spin groups
- Structure of $\mathrm{Spin}(n)$ influenced by unipotent elements

Results of Thesis

- Created algorithm that for every unipotent class C of $\mathrm{Spin}(n)$ determines Jordan normal form of $\mathrm{srep}_n(C)$. Based on:
  - Restriction Theorem $\rightarrow$ Induction
  - Knowledge of root subgroups $\rightarrow$ Exceptional class
- Derived some theoretical results

Thank you!

Appendix – Results with Original Labels

n	Class	Jordan blocks
1	(1)	1
2	(1^2)	1^2
3	(1^3)	1^2
	(3)	2
4	(1^4)	1^4
	$(1, 3)$	2^2
	$(2^2)_0$	$1^2, 2$
	$(2^2)_1$	$1^2, 2$

n	Class	Jordan blocks
5	(1^5)	1^4
	$(1^2, 3)$	2^2
	$(1, 2^2)$	$1^2, 2$
	(5)	4
6	(1^6)	1^8
	$(1^3, 3)$	2^4
	$(1^2, 2^2)$	$1^4, 2^2$
	$(1, 5)$	4^2
	(3^2)	$1^2, 3^2$

Appendix – Results in Dimension 9

Class	Jordan blocks	
	$\mathrm{char}(K) \neq 3$	$\mathrm{char}(K) = 3$
(1^9)	1^{16}	1^{16}
$(1^6, 3)$	2^8	2^8
$(1^5, 2^2)$	1^8, 2^4	1^8, 2^4
$(1^4, 5)$	4^4	4^4
$(1^3, 3^2)$	1^4, 3^4	1^4, 3^4
$(1^2, 2^2, 3)$	1^2, 2^4, 3^2	1^2, 2^4, 3^2
$(1^2, 7)$	1^2, 7^2	1^2, 7^2
$(1, 3, 5)$	3^2, 5^2	3^2, 5^2
$(1, 2^4)$	1^5, 2^4, 3	1^5, 2^4, 3
$(1, 4^2)$	1^3, 4^2, 5	1^3, 4^2, 5
$(2^2, 5)$	3, 4^2, 5	3, 4^2, 5
(3^3)	2^4, 4^2	2^2, 3^4
(9)	5, 11	7, 9

Appendix – Restriction Theorem

Theorem (A.)

Let n even. Then $\mathsf{srep}_n = \mathsf{srep}_n^+ \oplus \mathsf{srep}_n^-$ *where* srep_n^+ *and* srep_n^- *are irreducible, inequivalent and of the same dimension.*
Let $1 \leq r < n$ *and* $\beta\colon \mathrm{Spin}(r) \times \mathrm{Spin}(n-r) \to \mathrm{Spin}(n)$.
If r is odd, then

$$\mathsf{srep}_n^{\pm} \circ \beta = \mathsf{srep}_r \otimes \mathsf{srep}_{n-r}\,.$$

If r is even, then

$$\mathsf{srep}_n^{+} \circ \beta = (\mathsf{srep}_r^+ \oplus \mathsf{srep}_{n-r}^+) \otimes (\mathsf{srep}_r^- \oplus \mathsf{srep}_{n-r}^-),$$
$$\mathsf{srep}_n^{-} \circ \beta = (\mathsf{srep}_r^+ \oplus \mathsf{srep}_{n-r}^-) \otimes (\mathsf{srep}_r^- \oplus \mathsf{srep}_{n-r}^+).$$