

Cohen–Lenstra Heuristics: Distribution of Class Groups of Random Number Fields

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The Ideal Class Group

- K number field with ring of integers \mathcal{O}_K
- The **ideal class group** of K is

$$\text{Cl}_K := \frac{\{\text{fractional ideals of } K\}}{\{\text{principal fractional ideals of } K\}},$$

which is a finite abelian group

- Cl_K is **fundamental** object, measures how far \mathcal{O}_K is from being a UFD
- Given K explicitly, there are algorithms to compute Cl_K
- **Problem:** Not much known about structure of Cl_K in general

How to Understand Class Groups Better?

- Arithmetic Statistics: Study statistical behaviour of class groups in families of number fields
- Idea: (unknown) structure of class groups causes patterns, so **structural info** on class groups is tied to their **distribution**
- Thus if we understand their distribution, then we understand class groups a lot better
- Goal: Given G finite abelian, what is $\mathbb{P}(\text{Cl}_K \cong G)$ where K ranges over a certain family?

How to Understand Class Groups Better?

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Computational Data for Imaginary Quadratic Fields

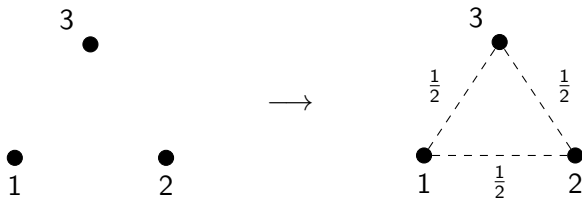
As $d < 0$ runs over squarefree integers...

- 3 divides $\#\text{Cl}_{\mathbb{Q}(\sqrt{d})}$ about 44% of the time,
 - the 3-Sylow subgroup $\text{Cl}_{\mathbb{Q}(\sqrt{d})}[3^\infty]$ is cyclic about 98% of the time.
-
- Data seems strange and not random. But e.g. $\#\text{Cl}_{\mathbb{Q}(\sqrt{d})}$ is not just a number, it is the size of a *group*!
 - What distribution should we even expect from **random groups**?

Excursion:
Distribution of Random Algebraic Objects

Excursion: Distribution of Random Graphs

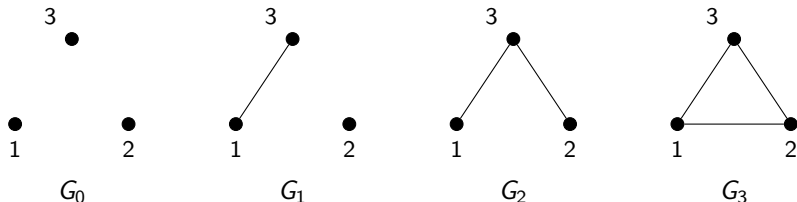
- Given 3 vertices, build a random graph R by independently inserting an edge between two vertices with probability $\frac{1}{2}$



- A graph isomorphism between two such graphs is $\sigma \in S_3$ such that vertices i and j are adjacent if and only if $\sigma(i)$ and $\sigma(j)$ are

Excursion: Distribution of Random Graphs

- Given 3 vertices, build a random graph R by independently inserting an edge between two vertices with probability $\frac{1}{2}$
- Possible outcomes up to isomorphism:



- Q: What are the probabilities $\mathbb{P}(R \cong G_j)$?

Excursion: Distribution of Random Graphs

- Each outcome has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$
- Let $\text{Iso}(G_i)$ be the set of graphs isomorphic to G_i , then

$$\mathbb{P}(R \cong G_i) = \frac{\#\text{Iso}(G_i)}{8}$$

- Nicer: S_3 operates transitively on $\text{Iso}(G_i)$ with stabiliser $\text{Aut}(G_i)$, so by orbit-stabiliser theorem $S_3/\text{Aut}(G_i) \xrightarrow{1:1} \text{Iso}(G_i)$, thus

$$\mathbb{P}(R \cong G_i) = \frac{6}{8} \cdot \frac{1}{\#\text{Aut}(G_i)}$$

Excursion: Distribution of Random Groups of Order n

- Generate a random group R of order n by writing down random $n \times n$ multiplication table (repeat if this is not a group structure)
- Q: If G is a group of order n , what is $\mathbb{P}(R \cong G)$?
- Exact same arguments as before yield

$$\mathbb{P}(R \cong G) = \frac{\#\{\text{tables isomorphic to } G\}}{\#\{\text{tables that give group structure}\}} \sim \frac{1}{\#\text{Aut}(G)}$$

Principle

The probability that a randomly chosen algebraic object is isomorphic to a given object G is proportional to $\frac{1}{\#\text{Aut}(G)}$.

Back to Class Groups

Cohen–Lenstra Heuristics for Imaginary Quadratic Fields

- Recall: Want to find distribution of Cl_K for K imaginary quadratic
- Look at distribution of p -Sylow subgroups $\text{Cl}_K[p^\infty]$ individually
- Computational data for $\text{Cl}_K[p^\infty]$ for odd p agrees with behaviour of random abelian p -groups!

Conjecture (Cohen–Lenstra, '83)

p odd prime, G finite abelian p -group. Then as K ranges over imaginary quadratic fields,

$$\mathbb{P}(\text{Cl}_K[p^\infty] \cong G) = \frac{c}{\#\text{Aut}(G)}$$

for a constant c depending only on p .

- Suggests that $\text{Cl}_K[p^\infty]$ does not carry additional structure!

Cohen–Lenstra Heuristics for Real Quadratic Fields

- For K real quadratic, $\text{Cl}_K[p^\infty]$ behaves differently; data suggests:

Conjecture (Cohen–Lenstra, '83)

p odd prime, G finite abelian p -group. Then as K ranges over real quadratic fields,

$$\mathbb{P}(\text{Cl}_K[p^\infty] \cong G) = \frac{c}{\#\text{Aut}(G) \cdot \#G}$$

for a constant c depending only on p .

- Non-random behaviour is related to \mathcal{O}_K^\times which now has rank 1
- Bartel–Lenstra (2020) conjecture that **Arakelov class group**, which knows about Cl_K and \mathcal{O}_K^\times , is random object as in our principle
- Principle guides us to a better behaved object

Bad Primes and Higher Degrees

- Conjectures have been extended to Galois extensions K/F for “good primes”
- There are “bad primes” for which distribution is not understood in many cases, including:
 - p that divide $|K : F|$
 - p for which $\mu_p \subseteq K$
- Overall: Cohen–Lenstra heuristics are **very strong conjectures** that would imply good understanding of class groups
- Many open questions, existing conjectures **vastly open!**